Preserving Log–Concavity Under Convolution: Comment

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In a recent work, Biais, Martimort, and Rochet (2000), BMR hereafter, develop a common agency model in which agents' types have two dimensions that lie on the real line and define a single dimensional aggregate:

$$\theta_0 = \theta_1 + \theta_2. \tag{1}$$

Thus, BMR face two alternative models of screening: either accounting for each source of asymmetry of information separately, *i.e.*, using $F_i(\theta_i)$, $i = 1, 2$, or targeting the aggregate directly using the convolution distribution:

$$F_0(\theta_0) = \int_{\Theta_j} F_i(\theta_0 - \theta_j) dF_j(\theta_j). \tag{2}$$

BMR assume that the distribution of θ_0 is log–concave in order to characterize a separating equilibrium in nonlinear schedules that depends exclusively on θ_0, thus reducing the dimensionality of the screening problem.

In arguing that their results are broadly applicable BMR claim that log–concavity of the convolution is not a very restrictive condition, since it is implied by the log–concavity of *either* the density of θ_1 or that of θ_2. They formally state this as Proposition 16 in the appendix to their paper and provide a 'proof'. Unfortunately, this result is false. Assume that $f_1(\theta_1)$ is the density function of a uniform random variable on the unit interval, and $f_2(\theta_2)$ is the density of a beta distribution with parameters $p = 0.4$ and $q = 0.5$, also defined on the unit interval. Both distributions are defined on bounded supports and the uniform density is log–concave as required by BMR. However, this beta density is not log–concave and thus, there are regions in the support of θ_0 where the uniform–beta convolution distribution $F_0(\theta_0)$ and the corresponding survival function $S_0(\theta) = 1 - F_0(\theta_0)$ are not log–concave.

Log–concave functions are PF_2, *i.e.*, *Pólya frequency functions* of order 2 [Karlin (1968, §7.1; Proposition 1.2)]. Any density function is at least PF_1 since $f_i(\theta_i) \geq 0$. The application of the *Basic Composition Formula* shows that the convolution of *Pólya frequency functions* of different order is also a *Pólya frequency function* of order equal to the lowest order of the convoluting densities [Karlin (1968, §1.2 and §3.1); Marshall and Olkin

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(1979, §18; A.4.a)]. Thus, the uniform–beta convolution of the previous counterexample is only PF_1 and log–concavity fails to be preserved. If *both* $f_1(\theta_1)$, and $f_2(\theta_2)$ are log–concave, the density function $f_0(\theta_0)$ is log–concave. Thus, $F_i(\theta_i)$ and $S_i(\theta_i)$ are also log–concave [Marshall and Olkin (1979, §18; B.1.a)] as BMR intended because the integral of log–concave functions is always log–concave [Prékopa (1973)]. All these results are not restricted to the case of distributions with bounded supports, and can be extended to discrete distributions as in Karlin (1968, §8).[2]

Nevertheless, to ensure the existence of a separating equilibrium in nonlinear strategies BMR only need the weaker assumption that $F_0(\theta_0)$ is increasing hazard rate (IHR). Assuming that the densities of θ_1 and θ_2 are both log–concave also leads to the result that $F_0(\theta_0)$ is IHR. But this approach reduces the set of distributions that can be used to model the screening of agents with stochastic demands because it excludes those IHR distributions whose density functions are not log–concave. A less restrictive approach is to ensure that the convolution distribution $F_0(\theta_0)$ is IHR if *both* convoluting distributions are IHR. The definition of the hazard rate function implies that any IHR distribution is characterized with a log–concave survival function [Marshall and Olkin (1979, §18; B.2.a)]. As survival functions are distribution functions themselves, preservation of the IHR property under convolution follows from the application of Theorem 5.3 of Karlin (1968, §3.5) that proves that the convolution of log–concave distributions is also log–concave regardless of whether their density functions are log–concave or not.

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[2] Log–concavity of density functions also ensures the preservation of the maximum likelihood ratio and single–peakedness properties. Miravete (2001b) discusses several economic applications of these results.